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The effect of g-jitter on double diffusion by natural convection from a sphere

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Abstract

This paper specifically considers the generation of steady streaming induced by g-jitter on double diffusion from a sphere immersed in a viscous and incompressible fluid. The governing equations of motion are first written in dimensionless forms and the resulting equations obtained after the introduction of the stream function are solved analytically and numerically. Analytical results using the matched asymptotic method are presented for the case when the Reynolds number, Re, is small ($Re \ll 1$), while numerical results using the Keller-box method are given for ($Re \gg 1$), or the boundary layer approximation. Both the cases of assisting and opposing thermal and concentration buoyancies are considered. Table and graphical results for the skin friction and heat and mass transfer from the sphere are presented and discussed for various parametric physical conditions. It is shown that for opposing buoyant forces the skin friction and heat and mass transfer rates follow complex trends depending on the buoyant ratio parameter, Prandtl and Schmidt numbers.
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Keywords: g-Jitter; Heat and mass transfer; Stokes layer; Boundary layer; Matched asymptotic expansion; Numerical results

1. Introduction

Various studies to investigate the effect of g-jitter, which is a term to describe fluctuating gravitational fields induced by machine vibrations and crew motions onboard a spacecraft have been addressed recently. For example, Alexander [1] carried out a numerical investigation on the effect of g-jitter on dopant concentration in a modeled crystal growth reactor. He concluded that low-frequency g-jitter can have a significant effect on dopant concentration. Li [2,3], Pan and Li [4], Suresh et al. [5] and Chamkha [6] reported analytical results for the g-jitter induced flows in microgravity under the influence of a transverse magnetic field for a simple system consisting of two vertical plates held at different temperatures. Results showed that the g-jitter frequency, applied magnetic field and temperature gradients all contribute to affect the convective flow. Rees and Pop [7–9] discussed g-jitter induced free convection effects in porous media and in viscous and incompressible fluids (non-porous media) were under the boundary layer approximation. Biringen and Danabasoglu [10] solved the full non-linear, time-dependent Boussinesq equations for g-jitter in a rectangular cavity. Their results showed the response to consist of a harmonic

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Nomenclature

a	radius of a sphere
C	concentration
C_∞	ambient concentration
g^*	g-jitter gravity field
g_0	magnitude of g-jitter
$\mathbf{k}$	unit vector pointing vertically upward
N	buoyancy ratio parameter
p	non-dimensional pressure
p_∞	ambient pressure
Pr	Prandtl number
q_w	wall heat flux
r	non-dimensional radial coordinate
Re	Reynolds number
t	non-dimensional time
T	non-dimensional fluid temperature
T_∞	ambient temperature
U_c	characteristic velocity
v_r, v_θ	non-dimensional velocity components along r and θ axes
$\mathbf{v}$	non-dimensional velocity vector

Greek symbols

β_C	concentration expansion coefficient
β_T	thermal expansion coefficient
θ	polar angle
η, $\bar{\eta}$	non-dimensional inner variables
υ	kinematic viscosity
ε	non-dimensional small quantity
ψ	non- dimensional stream function
ω^*	frequency of g-jitter oscillation

Superscripts

*	dimensional variables
′	differentiation with respect to $\bar{\eta}$
s	denotes steady part of the solution
u	denotes unsteady part of the solution

Subscripts

w	condition at the wall
∞	ambient condition

time-dependent component superposed over a steady streaming. The results of Farooq and Homsy [11,12] were complementary to those reported by Biringen and Danabasoglu [10], since by a weakly non-linear calculation, Farooq and Homsy [11,12] were able to explore parametric dependencies that explain physical mechanism and scaling.

All these studies have shed some light on the basic nature of g-jitter effects and have provided a thrust to devise useful mechanism by which the g-jitter induced convective flows may be suppressed. Also a fundamental understanding of some isolated aspects of fluid dynamic systems in an unsteady gravitational environment has been given. Given that perturbed accelerations exist in the microgravity environment, an estimation of the critical frequency ranges that drive a significant amount of convective motion, critical directions of modulation, and effects of random forcing have been estimated. Although these studies are useful in illustrating the basic features of g-jitter induced convection in a single component system, very little information seems to be available on the fundamental understanding of double diffusive convection in a microgravity environment [13].

In this paper, we consider the effect of g-jitter on the problem of double diffusion from an isothermal sphere that is immersed in a viscous and incompressible fluid. Double diffusive convection is referred to fluid flow generated by combined temperature and concentration gradients. It occurs in a wide range of scientific fields such as oceanography, astrophysics, geology, biology and chemical processes [14]. The study of double diffusion convection can be of critical importance in binary alloy solidification systems, because the quality of the final products is strongly correlated to the concentration distribution in the melt during processing.

Shu et al. [13] presented a numerical analysis of double diffusive convection induced by g-jitter in a cavity. Extensive simulations were carried out for temperature distribution and solutal (concentration) transport alloy system in space flights. The computations using finite element include the use of idealized single-frequency and multi-frequency g-jitter as well as real g-jitter data. These numerical results indicate that with an increase in g-jitter force (or amplitude), the non-linear convective effects become much more obvious, which in turn drastically change the concentration fields.

Our present work involves the generation of steady streaming for a double diffusion by natural convection from a sphere placed in a viscous and incompressible fluid under the influence of g-jitter of high frequency. The methodology here follows closely that of Amin [15] who investigated the heat transfer from a sphere immersed in an infinite fluid medium in a zero-gravity environment under the influence of g-jitter. Analytical results are presented for the case when the Reynolds number, Re, is small ($Re \ll 1$) and the Prandtl and Schmidt numbers are of O(1). Further, for the case ($Re \gg 1$), or the boundary layer approximation, and the Prandtl and Schmidt numbers are of O(1), numerical solutions are obtained using a very efficient implicit finite-difference method known as Keller-box method. The cases for assisting and opposing thermal and concentration buoyancies are also considered. The conclusion is that heat and mass transfer are negligible for high-frequency g-jitter but under special circumstances, when the Prandtl number and Schmidt numbers are high enough, low-frequency g-jitter may play an important role. It should be noticed that when the buoyancy forces due to the concentration are absent ($N = 0$), the solution of the heat transfer problem due to Amin [15] is recovered.

2. Basic equations

Consider the combined heat and mass transfer by natural convection from a fixed sphere of radius a immersed that is placed in a viscous and incompressible Boussinesq fluid, which is at uniform temperature, T_∞, and concentration, C_∞, respectively. We assume that the sphere is placed in a fluctuating gravitational field $g^*(t^*)\boldsymbol{k}$, where $\boldsymbol{k}$ is the unit vector pointing vertically upward, t^* is the time and we assume that $g^*(t^*) = g_0\cos(\omega^* t^*)$, where g_0 is the magnitude of the g-jitter and ω^* is the frequency of the g-jitter oscillation which is assumed very high ($\omega^* \gg 1$). It is also assumed that the sphere is subjected to a constant temperature, T_w ($>T_\infty$), and a constant concentration, C_w ($>C_\infty$), respectively. The g-jitter induced free convection is described by the continuity, Navier–Stokes, energy and concentration equations, which following Amin [15] can be written in non-dimensional form as

$$\nabla\cdot\boldsymbol{v} = 0 \tag{1}$$

$$\frac{\partial\boldsymbol{v}}{\partial t} + \varepsilon(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \frac{\varepsilon}{Re}\nabla^2\boldsymbol{v} + (T + NC)(\cos t)\boldsymbol{k} \tag{2}$$

$$\frac{\partial T}{\partial t} + \varepsilon(\boldsymbol{v}\cdot\nabla)T = \frac{\varepsilon}{PrRe}\nabla^2 T \tag{3}$$

$$\frac{\partial C}{\partial t} + \varepsilon(\boldsymbol{v}\cdot\nabla)C = \frac{\varepsilon}{ScRe}\nabla^2 C \tag{4}$$

where t is the non-dimensional time, $\boldsymbol{v}$ is the non-dimensional velocity vector, T is the non-dimensional fluid temperature, C is the non-dimensional concentration, p is the non-dimensional pressure, and Pr and Sc are the Prandtl and Schmidt numbers, respectively. The non-dimensional quantities are introduce in the form

$$\begin{aligned} &t = \omega t^*, \quad r = r^*/a, \quad \boldsymbol{v} = \boldsymbol{v}^*/U_c \\ &T = (T^* - T_\infty)/(T_w - T_\infty) \\ &C = (C^* - C_\infty)/(C_w - C_\infty) \\ &p = (p^* - p_\infty)/(\rho a\omega U_c) \end{aligned} \tag{5}$$

Further, U_c is the characteristic velocity, Re is the Reynolds number, ε is a dimensionless small parameter ($\varepsilon \ll 1$) and N is the buoyancy ratio, which are defined as

$$\begin{aligned} &U_c = g\beta_T(T_w - T_\infty)/\omega^*, \quad Re = U_c a/\upsilon, \quad \varepsilon = U_c/a\omega^* \\ &N = \beta_C(C_w - C_\infty)/\beta_T(T_w - T_\infty) \end{aligned} \tag{6}$$

with υ being the kinematic viscosity, and β_T and β_C are the coefficients of thermal and concentration expansions, respectively. With reference to spherical polar coordinates (r,θ,ϕ) with $\theta = 0$ corresponding to the direction of $\boldsymbol{k}$, we have for axisymmetric flow, $\boldsymbol{v} = (v_r, v_\theta, 0)$.

If we define the stream function ψ such that

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{7}$$

Eqs. (1)–(3) can then be written as

$$\frac{\partial}{\partial t}(D^2\psi) + \varepsilon\left\{\frac{1}{r^2}\frac{\partial(\psi, D^2\psi)}{\partial(r,\mu)} + \frac{2}{r^2}D^2\psi L_1\psi\right\} = \frac{\varepsilon}{Re}D^4\psi + (1-\mu^2)(L_2T + NL_2C)\cos t \tag{8}$$

$$\frac{\partial T}{\partial t} + \frac{\varepsilon}{r^2}\frac{\partial(\psi, T)}{\partial(r,\mu)} = \frac{\varepsilon}{RePr}\left(D^2T + \frac{2}{r^2}L_2T\right) \tag{9}$$

$$\frac{\partial C}{\partial t} + \frac{\varepsilon}{r^2}\frac{\partial(\psi, C)}{\partial(r,\mu)} = \frac{\varepsilon}{ReSc}\left(D^2C + \frac{2}{r^2}L_2C\right) \tag{10}$$

where

$$\begin{aligned} &D^2 = \frac{\partial^2}{\partial r} + \frac{(1-\mu^2)}{r^2}\frac{\partial^2}{\partial\mu^2}, \quad L_1 = \frac{\mu}{1-\mu^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\mu}, \\ &L_2 = r\frac{\partial}{\partial r} - \mu\frac{\partial}{\partial\mu} \end{aligned} \tag{11}$$

and $\mu = \cos\theta$. These equations are solved under the following boundary conditions

$$\psi = \frac{\partial\psi}{\partial r} = 0, \quad T = 1, \quad C = 1 \quad \text{on } r = 1 \tag{12a}$$

$$\psi = o(r^2), \quad T \to 0, \quad C \to 0 \quad \text{as } r \to \infty \tag{12b}$$

In order to solve Eqs. (8)–(10) subject to the boundary conditions (12), we shall follow the method of matched asymptotic method as used by Amin [15], namely, that the flow region is divided into an inner layer close to the sphere, where the viscous terms of Eq. (8) become important, and an outer layer, where the viscous terms of Eq. (8) are neglected, respectively. It is to be mentioned that the outer flow region is non-conservative, i.e. the flow is rotational in the present problem. Both the inner and outer flow and heat and mass transfer fields are determined simultaneously. The outer solutions satisfy the boundary conditions at infinity. The inner solutions, solved in a stretched coordinate, satisfy the boundary conditions on the sphere surface. These two solutions obtained in the inner and outer regions are then matched on the interface of these regions. All the unknowns will be determined by matching. From the usual boundary layer arguments it results in that the small non-dimensional parameter used to obtain solutions in this problem is ε/Re ($\ll 1$) because the thickness of the inner layer, or Stokes layer, is $O(\varepsilon/Re)^{1/2}$. It is worth pointing out that this method has been proposed by Amin [15] to study the heat transfer from an isothermal sphere induced by g-jitter when the buoyancy forces due to mass concentration are absent ($N = 0$). It has been shown by Amin [15] that the parameter ε/Re ($\ll 1$) captures very well the physics of the phenomena. Solutions for ($Re \ll 1$) with $Pr = O(1)$ and $Sc = O(1)$, and ($Re \gg 1$) with $Pr = O(1)$ and $Sc = O(1)$, respectively, will be considered.

3. Solution for $Re \ll 1$, $Pr = O(1)$ and $Sc = O(1)$

For the flow outside the boundary layers, or outer flow region that is rotational in this problem, the stream function ψ and temperature T and concentration C have to be expanded as

$$\chi = \chi_{00} + Re\chi_{01} + Re^2\chi_{02} + \cdots + \left(\frac{\varepsilon}{Re}\right)^{1/2}(\chi_{10} + Re\chi_{11} + Re^2\chi_{12}) + \cdots + \left(\frac{\varepsilon}{Re}\right)(\chi_{20} + Re\chi_{21} + Re^2\chi_{22} + \cdots) + \cdots \tag{13}$$

where χ denotes ψ, T or C. On the other hand, the variables for the inner layers are

$$\Psi = \left(\frac{Re}{2\varepsilon}\right)^{1/2}\psi, \quad \Im = T, \quad \Phi = C,$$

$$\eta = \left(\frac{Re}{2\varepsilon}\right)^{1/2}(r-1) \tag{14}$$

Substituting (14) into (8)–(10), we obtain equations for the inner layer, or Stokes layer, which can be written as

$$\frac{\partial}{\partial t}(D^2\Psi) + \varepsilon\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\left\{1 - 2\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta + \cdots\right\} \times \left\{\frac{\partial(\Psi, D^2\Psi)}{\partial(\eta,\mu)} + 2D^2\Psi L_3\Psi\right\} = \frac{1}{2}D^4\Psi + (1-\mu^2)\cos t L_4\Im \tag{15}$$

$$\frac{\partial\Im}{\partial t} + \varepsilon\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\left\{1 - 2\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta + \cdots\right\}\frac{\partial(\Psi,\Im)}{\partial(\eta,\mu)} = \frac{1}{2Pr}\left[D^2\Im + 2\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\left\{1 - 2\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta + \cdots\right\}L_4\Im\right] \tag{16}$$

$$\frac{\partial\Phi}{\partial t} + \varepsilon\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\left\{1 - 2\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta + \cdots\right\}\frac{\partial(\Psi,\Phi)}{\partial(\eta,\mu)} = \frac{1}{2Sc}\left[D^2\Phi + 2\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\left\{1 - 2\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta + \cdots\right\}L_4\Phi\right] \tag{17}$$

where

$$D^2 = \frac{\partial^2}{\partial\eta^2} + \left(\frac{2\varepsilon}{Re}\right)(1-\mu^2)\left\{1 - 2\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta + \cdots\right\}\frac{\partial^2}{\partial\mu^2}$$

$$L_3 = \frac{\mu}{1-\mu^2}\frac{\partial}{\partial\eta} + \left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\left\{1 - \left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta + \cdots\right\}\frac{\partial}{\partial\mu}$$

$$L_4 = \left\{1 + \left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta\right\}\frac{\partial}{\partial\eta} - \left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\mu\frac{\partial}{\partial\mu} \tag{18}$$

The expansions for Ψ, $\Im$ and Φ in the inner layer have the same form as in (13).

(i) *Solution at $O(Re^0)$*

To leading order, from Eqs. (8)–(10) and (15)–(17), we get the following equations for the outer and inner variables

$$\frac{\partial T_{00}}{\partial t} = 0, \quad \frac{\partial C_{00}}{\partial t} = 0 \tag{19a,b}$$

$$\frac{\partial D^2\psi_{00}}{\partial t} = (1-\mu^2)(L_2T_{00} + NL_2C_{00})\cos t \tag{19c}$$

$$\frac{\partial\Im_{00}}{\partial t} = \frac{1}{2Pr}\frac{\partial^2\Im_{00}}{\partial\eta^2}, \quad \frac{\partial\Phi_{00}}{\partial t} = \frac{1}{2Sc}\frac{\partial^2\Phi_{00}}{\partial\eta^2} \tag{20a,b}$$

$$\frac{\partial D^2\Psi_{00}}{\partial t} = \frac{1}{2}\frac{\partial^4\Psi_{00}}{\partial\eta^4} + (1-\mu^2)\left(\frac{\partial\Im_{00}}{\partial\eta} + N\frac{\partial\Phi_{00}}{\partial\eta}\right)\cos t \tag{20c}$$

The solutions of Eqs. (20) which satisfy the boundary conditions (12a) are

$$\Im_{00}^{(s)} = 1 + a_{00}\eta, \quad \Phi_{00}^{(s)} = 1 + b_{00}\eta \tag{21}$$

$$\Psi_{00}^{(u)} = c_{00}(1+i)\left\{\eta - \frac{(1-i)}{2}[1 - e^{-(1+i)\eta}]\right\}(1-\mu^2)e^{it} \tag{22}$$

where $i = \sqrt{-1}$ is the imaginary unity. Here and in what follows, superscripts (s) and (u) are used to indicate the time-independent, or steady, and time-dependent, or unsteady components of the solution at each stage. In (21) it has been anticipated that the temperature and concentration will be steady at leading order, as dictated by the boundary condition (12a), so that the flow driven by the fluctuating gravitational field will be unsteady. We notice that the right-hand side of Eq. (19c) cannot be fully determined at this stage because Eqs. (19a,b) merely indicate that $T_{00} = T_{00}^{(s)}$ and $C_{00} = C_{00}^{(s)}$. To determine $T_{00}^{(s)}$ and $C_{00}^{(s)}$ we need to consider terms of relative order ε/Re in Eqs. (9) and (10). Thus, from the terms of $O(\varepsilon/Re)$, we have

$$\frac{\partial T_{20}}{\partial t} = \frac{1}{Pr}\left(D^2T_{00}^{(s)} + \frac{2}{r^2}L_2T_{00}^{(s)}\right) \tag{23}$$

$$\frac{\partial C_{20}}{\partial t} = \frac{1}{Sc}\left(D^2C_{00}^{(s)} + \frac{2}{r^2}L_2C_{00}^{(s)}\right) \tag{24}$$

If we now separate the unsteady and steady parts of (23) and (24), we have as the equations for $T_{00}^{(s)}$ and $C_{00}^{(s)}$,

$$D^2T_{00}^{(s)} + \frac{2}{r^2}L_2T_{00}^{(s)} = 0 \tag{25}$$

$$D^2C_{00}^{(s)} + \frac{2}{r^2}L_2C_{00}^{(s)} = 0 \tag{26}$$

Matching the solutions of (25) and (26) with (21) requires $a_{00} = b_{00} = 0$ and $T_{00}^{(s)} = C_{00}^{(s)} = \frac{1}{r}$. If we use further the matching principle for Eqs. (19c) and (22), we get $c_{00} = -(1+\mathrm{i})(1+N)/2$, so that we have at O(1),

$$\psi_{00}^{(u)} = -\frac{\mathrm{i}}{2}(1-\mu^2)(1+N)\left(r - \frac{1}{r}\right)\mathrm{e}^{\mathrm{i}t}$$

$$\Psi_{00}^{(u)} = -\mathrm{i}\left\{\eta - \frac{(1-\mathrm{i})}{2}[1-\mathrm{e}^{-(1+\mathrm{i})\eta}]\right\}(1-\mu^2)(1+N)\mathrm{e}^{\mathrm{i}t},$$

$$\Psi_{00}^{(s)} = \psi_{00}^{(s)} = 0, \quad T_{00}^{(s)} = \frac{1}{r}, \quad \mathfrak{J}_{00}^{(s)} = 1,$$

$$C_{00}^{(s)} = \frac{1}{r}, \quad \Phi_{00}^{(s)} = 1 \tag{27}$$

(ii) *Solution at O(Re)*

We notice from (27) that there is no time-independent flow induced at this leading order. With the solution at the leading order complete, we next consider the term O(*Re*) in the series (13). The corresponding equations for O(*Re*) terms are

$$\frac{\partial T_{01}}{\partial t} = 0 \tag{28}$$

$$\frac{\partial C_{01}}{\partial t} = 0 \tag{29}$$

$$\frac{\partial D^2\psi_{01}}{\partial t} = (1-\mu^2)(L_2T_{01} + NL_2C_{01})\cos t \tag{30}$$

for the outer layer and

$$\frac{\partial \mathfrak{J}_{01}}{\partial t} = \frac{1}{2Pr}\frac{\partial^2\mathfrak{J}_{01}}{\partial\eta^2} \tag{31}$$

$$\frac{\partial \Phi_{01}}{\partial t} = \frac{1}{2Sc}\frac{\partial^2\Phi_{01}}{\partial\eta^2} \tag{32}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\Psi_{01}}{\partial\eta^2}\right) = \frac{1}{2}\frac{\partial^4\Psi_{01}}{\partial\eta^4} + (1-\mu^2)\left(\frac{\partial\mathfrak{J}_{01}}{\partial\eta} + N\frac{\partial\Phi_{01}}{\partial\eta}\right)\cos t \tag{33}$$

for the inner layer, respectively.

Eqs. (28) and (29) imply that $T_{01} = T_{01}^{(s)}$ and $C_{01} = C_{01}^{(s)}$, while (30) provides an equation for $\psi_{01}^{(u)}$. To complete the right-hand side of (30), we consider the term of relative order $\mathrm{O}((\varepsilon/Re)Re)$ in Eqs. (9) and (10). Thus, the term $\mathrm{O}(\varepsilon)$ gives

$$\frac{\partial T_{21}}{\partial t} + \frac{1}{r^2}\left(\frac{\partial\psi_{00}}{\partial r}\frac{\partial T_{00}}{\partial\mu} - \frac{\partial\psi_{00}}{\partial\mu}\frac{\partial T_{00}}{\partial r}\right) = \frac{1}{Pr}\left(D^2T_{01} + \frac{2}{r^2}L_2T_{01}\right) \tag{34}$$

$$\frac{\partial C_{21}}{\partial t} + \frac{1}{r^2}\left(\frac{\partial\psi_{00}}{\partial r}\frac{\partial C_{00}}{\partial\mu} - \frac{\partial\psi_{00}}{\partial\mu}\frac{\partial C_{00}}{\partial r}\right) = \frac{1}{Sc}\left(D^2C_{01} + \frac{2}{r^2}L_2C_{01}\right) \tag{35}$$

As before, the equations for T_{01} and C_{01} become, after we separate the steady and unsteady components of Eqs. (34) and (35),

$$D^2T_{01}^{(s)} + \frac{2}{r^2}L_2T_{01}^{(s)} = 0 \tag{36}$$

$$D^2C_{01}^{(s)} + \frac{2}{r^2}L_2C_{01}^{(s)} = 0 \tag{37}$$

Solutions of Eqs. (31) and (32) for the inner region care simply $\mathfrak{J}_{01}^{(s)} = a_{01}\eta$ and $\Phi_{01}^{(s)} = b_{01}\eta$, where a_{01} and b_{01} are constants and matching it with the outer solutions give $\mathfrak{J}_{01}^{(s)} = a_{01} = 0$ and $\Phi_{01}^{(s)} = b_{01} = 0$. It follows immediately from (36) and (37) that

$$T_{01}^{(s)} = C_{01}^{(s)} = 0 \tag{38}$$

The equations for $\psi_{01}^{(u)}$ and $\Psi_{01}^{(u)}$ are now

$$D^2\psi_{01}^{(u)} = 0 \tag{39}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\Psi_{01}^{(u)}}{\partial\eta^2}\right) - \frac{1}{2}\left(\frac{\partial^4\Psi_{01}^{(u)}}{\partial\eta^4}\right) = 0 \tag{40}$$

and the solution of these equations, with the homogeneous boundary conditions (12), is simply

$$\psi_{01}^{(u)} = \Psi_{01}^{(u)} = 0 \tag{41}$$

To complete the solution at the order O(*Re*), we must next consider the steady parts of ψ_{01} and Ψ_{01}. From (33) and the boundary condition (12a), we have

$$\Psi_{01}^{(s)} = (A\eta^3 + B\eta^2)\mu(1-\mu^2)(1+N)^2 \tag{42}$$

where A and B are unknown constants yet.

The equation for $\psi_{01}^{(s)}$ follows by considering, as with the energy and concentration equations, which are of relative order $(\varepsilon/Re)Re$. Thus, the terms $\mathrm{O}(\varepsilon)$ in Eq. (8) gives

$$-\frac{\partial}{\partial t}(D^2\psi_{21}) + D^4\psi_{01} = -(1-\mu^2)(L_2T_{21} + NL_2C_{21})\cos t + \frac{1}{r^2}\left(\frac{\partial\psi_{00}^{(u)}}{\partial r}\frac{\partial D^2\psi_{00}^{(u)}}{\partial\mu} - \frac{\partial\psi_{00}^{(u)}}{\partial\mu}\frac{\partial D^2\psi_{00}^{(u)}}{\partial r}\right) + \frac{2}{r^2}D^2\psi_{00}^{(u)}L_1\psi_{00}^{(u)} \tag{43}$$

Because $\psi_{00} \propto \sin t$, it is clear that the right-hand side of Eq. (43) includes terms that are independent of t in addition to the higher harmonic cos $2t$. This is also true of the buoyancy term in Eq. (43) because the time-dependent component of (34) and (35), namely,

$$\frac{\partial T_{21}}{\partial t} = -\frac{1}{r^2}\left(\frac{\partial \psi_{00}^{(u)}}{\partial r}\frac{\partial T_{00}^{(s)}}{\partial \mu} - \frac{\partial \psi_{00}^{(u)}}{\partial \mu}\frac{\partial T_{00}^{(s)}}{\partial r}\right) \tag{44}$$

$$\frac{\partial C_{21}}{\partial t} = -\frac{1}{r^2}\left(\frac{\partial \psi_{00}^{(u)}}{\partial r}\frac{\partial C_{00}^{(s)}}{\partial \mu} - \frac{\partial \psi_{00}^{(u)}}{\partial \mu}\frac{\partial C_{00}^{(s)}}{\partial r}\right) \tag{45}$$

gives, on integration with respect to t,

$$\begin{aligned} T_{21} &= \left(\frac{1}{r^5} - \frac{1}{r^3}\right)\mu(1+N)\cos t + T_{21}^{(s)}(r,\mu) \\ C_{21} &= \left(\frac{1}{r^5} - \frac{1}{r^3}\right)\mu(1+N)\cos t + C_{21}^{(s)}(r,\mu) \end{aligned} \tag{46}$$

It is of interest to note that because the fluctuating body force is non-conservative, the dominant part of the fluctuating flow field, ψ_{00}, is rotational, with the consequence that the steady streaming at $O(Re)$, namely, $\psi_{01}^{(s)}$, is induced in part by the direct action of the Reynolds stresses in the flow outside the Stokes shear layer. This may be contrasted with the classical studies in which a body vibrates in a fluid at rest, (see [16,17]), or the fluctuating heat transfer of Merkin [18] and Davidson [19], where the outer flow is non-rotational and the weaker steady streaming is induced indirectly from the action of Reynolds stresses within the Stokes layer. This element of the steady streaming is recovered at $O(\varepsilon)$ in this study.

If we separate the time-independent and time-dependent parts of Eq. (43) we obtain an equation for $\psi_{01}^{(s)}$ with the solution

$$\psi_{01}^{(s)} = \left(\frac{A_{01}}{r^2} - B_{01} - \frac{1}{16r} - \frac{1}{48}r\right)\mu(1-\mu^2)(1+N)^2 \tag{47}$$

where A_{01} and B_{01} are constants. Matching the inner and the outer solutions now gives for $A_{01} = 1/48$, $B_{01} = 3/48$ and in (42), $A = B = 0$. Thus, at $O(Re)$, we have

$$\begin{aligned} \psi_{01}^{(s)} &= \frac{1}{48}\left(\frac{1}{r^2} - \frac{3}{r} + 3 - r\right)\mu(1-\mu^2)(1+N)^2, \\ \Psi_{01}^{(s)} &= 0 \end{aligned} \tag{48}$$

(iii) *Solution at $O(Re^2)$*

At this order of Re, we only consider the equations for the temperature and concentration in both the outer and inner regions, where the equation

$$\frac{\partial T_{02}}{\partial t} = 0 \tag{49}$$

$$\frac{\partial C_{02}}{\partial t} = 0 \tag{50}$$

in the outer region implies $T_{02} = T_{02}^{(s)}$ and $C_{02} = C_{02}^{(s)}$, and in the inner region the equations for $\Im_{02}$ and Φ_{02} are

$$\frac{\partial \Im_{02}}{\partial t} - \frac{1}{2Pr}\frac{\partial^2 \Im_{02}}{\partial \eta^2} = 0 \tag{51}$$

$$\frac{\partial \Phi_{02}}{\partial t} - \frac{1}{2Sc}\frac{\partial^2 \Phi_{02}}{\partial \eta^2} = 0 \tag{52}$$

The equation satisfied by $T_{02}^{(s)}$ and $C_{02}^{(s)}$ as at the earlier stages, obtained when we consider the terms of relative order $O((\varepsilon/Re)Re^2)$ in Eqs. (9) and (10). Thus, the terms $O(\varepsilon Re)$

$$\begin{aligned} &\frac{\partial T_{22}}{\partial t} + \frac{1}{r^2}\left(\frac{\partial \psi_{01}}{\partial r}\frac{\partial T_{00}^{(s)}}{\partial \mu} - \frac{\partial \psi_{01}}{\partial \mu}\frac{\partial T_{00}^{(s)}}{\partial r}\right) \\ &\quad = \frac{1}{Pr}\left(D^2 T_{02}^{(s)} + \frac{2}{r^2}L_2 T_{02}^{(s)}\right) \end{aligned} \tag{53}$$

$$\begin{aligned} &\frac{\partial C_{22}}{\partial t} + \frac{1}{r^2}\left(\frac{\partial \psi_{01}}{\partial r}\frac{\partial C_{00}^{(s)}}{\partial \mu} - \frac{\partial \psi_{01}}{\partial \mu}\frac{\partial C_{00}^{(s)}}{\partial r}\right) \\ &\quad = \frac{1}{Sc}\left(D^2 C_{02}^{(s)} + \frac{2}{r^2}L_2 C_{02}^{(s)}\right) \end{aligned} \tag{54}$$

The steady part of Eqs. (53) and (54) gives the equations for $T_{02}^{(s)}$ and $C_{02}^{(s)}$ as

$$\begin{aligned} &\frac{1}{Pr}\left(D^2 T_{02}^{(s)} + \frac{2}{r^2}L_2 T_{02}^{(s)}\right) \\ &\quad = \frac{1}{r^2}\left(\frac{\partial \psi_{01}^{(s)}}{\partial r}\frac{\partial T_{00}^{(s)}}{\partial \mu} - \frac{\partial \psi_{01}^{(s)}}{\partial \mu}\frac{\partial T_{00}^{(s)}}{\partial r}\right) \end{aligned} \tag{55}$$

$$\begin{aligned} &\frac{1}{Sc}\left(D^2 C_{02}^{(s)} + \frac{2}{r^2}L_2 C_{02}^{(s)}\right) \\ &\quad = \frac{1}{r^2}\left(\frac{\partial \psi_{01}^{(s)}}{\partial r}\frac{\partial C_{00}^{(s)}}{\partial \mu} - \frac{\partial \psi_{01}^{(s)}}{\partial \mu}\frac{\partial C_{00}^{(s)}}{\partial r}\right) \end{aligned} \tag{56}$$

whose solutions are given by

$$\begin{aligned} T_{02}^{(s)} &= \left\{\frac{A_{02}}{r^3} + \frac{Pr}{48}\left(\frac{1}{6r^4} - \frac{3}{4r^2} + \frac{1}{6r} + \frac{3}{5r^3}\ln r\right)\right\} \\ &\quad \times (1-3\mu^2)(1+N)^2 \end{aligned} \tag{57}$$

$$\begin{aligned} C_{02}^{(s)} &= \left\{\frac{B_{02}}{r^3} + \frac{Sc}{48}\left(\frac{1}{6r^4} - \frac{3}{4r^2} + \frac{1}{6r} + \frac{3}{5r^3}\ln r\right)\right\} \\ &\quad \times (1-3\mu^2)(1+N)^2 \end{aligned} \tag{58}$$

where A_{02} and B_{02} are unknown constants yet.

In the inner region, the solutions for $\Im_{02}$ and Φ_{02} are $a_{02}(1-3\mu^2)(1+N)^2$ and $b_{02}(1-3\mu^2)(1+N)^2$ where a_{02} and b_{02} are constants, which when matched with outer solutions (57) and (58) give $a_{02} = b_{02} = 0$, $A_{02} = 5Pr/576$ and $B_{02} = 5Sc/576$, so that $\Im_{02}^{(s)} = \Phi_{02}^{(s)} = 0$, and $T_{02}^{(s)}$ and $C_{02}^{(s)}$ have the expressions

$$\begin{aligned} T_{02}^{(s)} &= Pr\left(\frac{1}{288r} - \frac{1}{64r^2} + \frac{5}{576r^3} + \frac{1}{80r^3}\ln r + \frac{1}{288r^4}\right) \\ &\quad \times (1-3\mu^2)(1+N)^2 \end{aligned} \tag{59}$$

$$\begin{aligned} C_{02}^{(s)} &= Sc\left(\frac{1}{288r} - \frac{1}{64r^2} + \frac{5}{576r^3} + \frac{1}{80r^3}\ln r + \frac{1}{288r^4}\right) \\ &\quad \times (1-3\mu^2)(1+N)^2 \end{aligned} \tag{60}$$

We notice that it is only at this stage that convective heat transfer influences the time-independent part of the temperature and concentration fields.

In terms of our expansion parameter $(\varepsilon/Re)^{1/2}$, the procedure for determining the stream function, temperature and concentration at $O(\varepsilon/Re)^{1/2}$ in both the inner and outer regions follows a similar pattern to that for obtaining the solutions at $O(\varepsilon/Re)^0$, which we have described in detail above. Accordingly, the solutions obtained at the orders $O(\varepsilon/Re)^{1/2}$, $O((\varepsilon/Re)^{1/2}Re)$ and $O((\varepsilon/Re)^{1/2}Re^2)$, respectively, are summarized below:

$$\psi_{10}^{(s)} = \Psi_{10}^{(s)} = T_{10} = \Im_{10}^{(u)} = C_{10} = \Phi_{10}^{(u)} = 0$$

$$\psi_{10}^{(u)} = \frac{\sqrt{2}}{2r}(1-\mathrm{i})(1-\mu^2)(1+N)\mathrm{e}^{\mathrm{i}t}$$

$$\Psi_{10}^{(u)} = -\frac{1}{\sqrt{2}}[1-(1+\mathrm{i})\eta - \mathrm{e}^{-(1+\mathrm{i})\eta} - \mathrm{i}\eta^2](1-\mu^2)(1+N)\mathrm{e}^{\mathrm{i}t}$$

$$\Im_{10}^{(s)} = \Phi_{10}^{(s)} = -\sqrt{2}\eta \tag{61}$$

$$\psi_{11}^{(u)} = \Psi_{11} = T_{11} = \Im_{11} = C_{11} = \Phi_{11} = 0$$

$$\psi_{11}^{(s)} = \frac{\sqrt{2}}{32}(1-\mathrm{i})\left(\frac{1}{r^2} + 1 - \frac{2}{r}\right)\mu(1-\mu^2)(1+N)^2 \tag{62}$$

$$T_{12}^{(u)} = \Im_{12}^{(u)} = C_{12}^{(u)} = \Phi_{12}^{(u)} = 0$$

$$T_{12}^{(s)} = \frac{\sqrt{2}Pr}{32}(1-\mathrm{i})\left(\frac{1}{12r^3} + \frac{1}{6r^4} - \frac{1}{4r^2} + \frac{2}{5r^3}\ln r\right) \times (1-3\mu^2)(1+N)^2$$

$$C_{12}^{(s)} = \frac{\sqrt{2}Sc}{32}(1-\mathrm{i})\left(\frac{1}{12r^3} + \frac{1}{6r^4} - \frac{1}{4r^2} + \frac{2}{5r^3}\ln r\right) \times (1-3\mu^2)(1+N)^2 \tag{63}$$

$$\Im_{12}^{(s)} = \frac{\sqrt{2}Pr}{2880}\eta(1-3\mu^2)(1+N)^2$$

$$\Phi_{12}^{(s)} = \frac{\sqrt{2}Sc}{2880}\eta(1-3\mu^2)(1+N)^2$$

We remark that the first non-zero perturbation to the temperature and concentration in the outer region in Eqs. (61)–(63), namely $T_{12}^{(s)}$ and $C_{12}^{(s)}$ again arises from steady convective effects. The solutions obtained at $O(\varepsilon/Re)$ is

$$\psi_{20}^{(s)} = \Psi_{20}^{(s)} = T_{20} = \Im_{20}^{(u)} = C_{20} = \Phi_{20}^{(u)} = 0$$

$$\Im_{20}^{(s)} = \Phi_{20}^{(s)} = 2\eta^2$$

$$\psi_{20}^{(u)} = \frac{1}{r}(1-\mu^2)(1+N)\mathrm{e}^{\mathrm{i}t}$$

$$\Psi_{20}^{(u)} = [-\eta + (1+\mathrm{i})\eta^2 - \mathrm{i}\eta^3 + \eta\mathrm{e}^{-(1+\mathrm{i})\eta}](1-\mu^2)(1+N)\mathrm{e}^{\mathrm{i}t} \tag{64}$$

We have further determined the solutions at $O((\varepsilon/Re)Re)$ and $O((\varepsilon/Re)Re^2)$, respectively, but we will not give here these solutions since their expressions are rather long. It is, however, worth pointing out that up to $O(\varepsilon/Re)$ the Stokes shear-wave layer had no role to play with respect to $\Psi^{(s)}$. However, the equation satisfied by $\Psi_{21}^{(s)}$ is non-homogeneous and in particular includes a contribution from the Reynolds stresses, which act in the Stokes layer. In the classical studies, and earlier oscillatory heat transfer studies it is the streaming induced in the Stokes layer that is wholly responsible for the time-independent part of motion. As we have already noted above, the Reynolds stresses, which act in the outer region provide a more significant contribution to the steady streaming in this problem.

4. Solution for $Re \gg 1$, $Pr = O(1)$ and $Sc = O(1)$

Proceeding in a similar way as for the case when $Re \ll 1$, we can obtain equations for the functions $\psi_0^{(s)}$, $\psi_0^{(u)}$, $T_0^{(s)}$, $T_2^{(u)}$, $C_0^{(s)}$ and $C_2^{(u)}$ when $Re \gg 1$ in the following form, see [15] for further details,

$$\frac{\partial}{\partial t}(D^2\psi_0^{(u)}) = (1-\mu^2)(L_2T_0^{(s)} + NL_2C_0^{(s)})\cos t \tag{65}$$

$$\frac{1}{Re}D^4\psi_0^{(s)} - \left\{\frac{1}{r^2}\frac{\partial(\psi_0^{(s)}, D^2\psi_0^{(s)})}{\partial(r,\mu)} + \frac{2}{r^2}D^2\psi_0^{(s)}L_1\psi_0^{(s)}\right\} = \frac{1}{r^2}\frac{\partial(\psi_0^{(u)}, D^2\psi_0^{(u)})}{\partial(r,\mu)} + \frac{2}{r^2}D^2\psi_0^{(u)}L_1\psi_0^{(u)} - \frac{(1-\mu^2)}{Re}(L_2T_2^{(u)} + NL_2C_2^{(u)})\cos t \tag{66}$$

$$\frac{\partial T_2^{(u)}}{\partial t} = -\frac{Re}{r^2}\frac{\partial(\psi_0^{(u)}, T_0^{(s)})}{\partial(r,\mu)}, \quad \frac{\partial C_2^{(u)}}{\partial t} = -\frac{Re}{r^2}\frac{\partial(\psi_0^{(u)}, C_0^{(s)})}{\partial(r,\mu)} \tag{67}$$

$$\frac{1}{Pr}\left(D^2T_0^{(s)} + \frac{2}{r^2}L_2T_0^{(s)}\right) = \frac{Re}{r^2}\frac{\partial(\psi_0^{(s)}, T_0^{(s)})}{\partial(r,\mu)} \tag{68}$$

$$\frac{1}{Sc}\left(D^2C_0^{(s)} + \frac{2}{r^2}L_2C_0^{(s)}\right) = \frac{Re}{r^2}\frac{\partial(\psi_0^{(s)}, C_0^{(s)})}{\partial(r,\mu)} \tag{69}$$

It should be noticed that the right-hand side of Eq. (66) consists of the contribution of the Reynolds-stress and buoyancy due to thermal and concentration terms. This is situation is in contrast to the classical one, in which a steady streaming is induced by vibrations of a solid body in a viscous fluid at rest or with that of free convection from a circular cylinder whose surface temperature oscillates about a mean ambient temperature in a constant gravitational field. In these situations the dominant fluctuating flow is non-rotational while in the present problem this fluctuating flow is rotational.

We shall consider now the limiting case when $Re \gg 1$, or boundary layer approximation for the steady streaming flow. This boundary layer has the thickness $O(Re^{-1/2})$, and encompasses the much thinner Stokes layer for $Re \ll 1$. The variables appropriate to this boundary layer region are

$$\psi = Re^{-1/2}\overline{\psi}(t,\mu,\overline{\eta}), \quad T = \overline{\Im}(t,\mu,\overline{\eta}),$$
$$C = \overline{\Phi}(t,\mu,\overline{\eta}), \quad r-1 = Re^{-1/2}\overline{\eta} \tag{70}$$

Substituting (70) into Eqs. (65)–(69) and letting $Re \to \infty$, we obtain the following g-jitter driven boundary layer equations for the corresponding functions $\overline{\psi}_0^{(s)}$, $\overline{\psi}_0^{(u)}$, $\overline{\Im}_0^{(s)}$, $\overline{\Phi}_0^{(s)}$, $\overline{\Im}_2^{(u)}$ and $\overline{\Phi}_2^{(u)}$,

$$\frac{\partial^2\overline{\psi}_0^{(u)}}{\partial\overline{\eta}^2} = (1-\mu^2)\left(\frac{\partial\overline{\Im}_0^{(s)}}{\partial\overline{\eta}} + N\frac{\partial\overline{\Phi}_0^{(s)}}{\partial\overline{\eta}}\right)\sin t \tag{71}$$

$$\begin{aligned}&\frac{\partial^4\overline{\psi}_0^{(s)}}{\partial\overline{\eta}^4} - \frac{\partial(\overline{\psi}_0^{(s)}, \partial^2\overline{\psi}_0^{(s)}/\partial\overline{\eta}^2)}{\partial(\overline{\eta},\mu)} - \frac{2\mu}{1-\mu^2}\frac{\partial^2\overline{\psi}_0^{(s)}}{\partial\overline{\eta}^2}\frac{\partial\overline{\psi}_0^{(s)}}{\partial\overline{\eta}}\\
&= \left.\frac{\partial(\overline{\psi}_0^{(u)}, \partial^2\overline{\psi}_0^{(u)}/\partial\overline{\eta}^2)}{\partial(\overline{\eta},\mu)}\right|^{(s)} + \left.\frac{2\mu}{1-\mu^2}\frac{\partial^2\overline{\psi}_0^{(u)}}{\partial\overline{\eta}^2}\frac{\partial\overline{\psi}_0^{(u)}}{\partial\overline{\eta}}\right|^{(s)}\\
&\quad - \left.\frac{(1-\mu^2)}{Re}\cos t\left(\frac{\partial\overline{\Im}_2^{(u)}}{\partial\overline{\eta}} + N\frac{\partial\overline{\Phi}_2^{(u)}}{\partial\overline{\eta}}\right)\right|^{(s)}\end{aligned} \tag{72}$$

$$\frac{\partial\overline{\Im}_2^{(u)}}{\partial t} = -Re\frac{\partial(\overline{\psi}_0^{(u)}, \overline{\Im}_0^{(s)})}{\partial(\overline{\eta},\mu)} \tag{73}$$

$$\frac{\partial\overline{\Phi}_2^{(u)}}{\partial t} = -Re\frac{\partial(\overline{\psi}_0^{(u)}, \overline{\Phi}_0^{(s)})}{\partial(\overline{\eta},\mu)} \tag{74}$$

$$\frac{1}{Pr}\frac{\partial\overline{\Im}_0^{(s)}}{\partial\overline{\eta}^2} = \frac{\partial(\overline{\psi}_0^{(s)}, \overline{\Im}_0^{(s)})}{\partial(\overline{\eta},\mu)} \tag{75}$$

$$\frac{1}{Sc}\frac{\partial\overline{\Phi}_0^{(s)}}{\partial\overline{\eta}^2} = \frac{\partial(\overline{\psi}_0^{(s)}, \overline{\Phi}_0^{(s)})}{\partial(\overline{\eta},\mu)} \tag{76}$$

To obtain an equation for the steady streaming function $\overline{\psi}_0^{(s)}$, we eliminate $\overline{\psi}_0^{(u)}$, $\overline{\Im}_2^{(u)}$ and $\overline{\Phi}_2^{(u)}$ from Eqs. (71)–(74) as follows. Eq. (71) is integrated twice with respect to $\overline{\eta}$ to give

$$\overline{\psi}_0^{(u)} = (1-\mu^2)\sin t\int_0^{\overline{\eta}}(\overline{\Im}_0^{(s)}(x,\mu) + N\overline{\Phi}_0^{(s)}(x,\mu))\,\mathrm{d}x \tag{77}$$

Substituting this relation into Eqs. (73) and (74), followed by an integration with respect to t, we obtain, after some algebra,

$$\begin{aligned}&(\cos t)^{-1}\overline{\Im}_2^{(u)}\\
&= (1-\mu^2)\overline{\Im}_0^{(s)}\left(\frac{\partial\overline{\Im}_0^{(s)}}{\partial\mu} + N\frac{\partial\overline{\Phi}_0^{(s)}}{\partial\mu}\right)\\
&\quad + 2\mu\frac{\partial\overline{\Im}_0^{(s)}}{\partial\overline{\eta}}\int_0^{\overline{\eta}}(\overline{\Im}_0^{(s)}(x,\mu) + N\overline{\Phi}_0^{(s)})\,\mathrm{d}x\\
&\quad - (1-\mu^2)\frac{\partial\overline{\Im}_0^{(s)}}{\partial\overline{\eta}}\int_0^{\overline{\eta}}\left(\frac{\partial\overline{\Im}_0^{(s)}(x,\mu)}{\partial\mu} + N\frac{\partial\overline{\Phi}_0^{(s)}(x,\mu)}{\partial\mu}\right)\mathrm{d}x\end{aligned} \tag{78}$$

$$\begin{aligned}&(\cos t)^{-1}\overline{\Phi}_2^{(u)}\\
&= (1-\mu^2)\overline{\Im}_0^{(s)}\left(\frac{\partial\overline{\Im}_0^{(s)}}{\partial\mu} + N\frac{\partial\overline{\Phi}_0^{(s)}}{\partial\mu}\right)\\
&\quad + 2\mu\frac{\partial\overline{\Im}_0^{(s)}}{\partial\overline{\eta}}\int_0^{\overline{\eta}}(\overline{\Im}_0^{(s)}(x,\mu) + N\overline{\Phi}_0^{(s)})\,\mathrm{d}x\\
&\quad - (1-\mu^2)\frac{\partial\overline{\Im}_0^{(s)}}{\partial\overline{\eta}}\int_0^{\overline{\eta}}\left(\frac{\partial\overline{\Im}_0^{(s)}(x,\mu)}{\partial\mu} + N\frac{\partial\overline{\Phi}_0^{(s)}(x,\mu)}{\partial\mu}\right)\mathrm{d}x\end{aligned} \tag{79}$$

We now integrate Eq. (72) once with respect to $\overline{\eta}$ and use Eqs. (71), (77), (78) and (79) to obtain the following boundary layer equation for $\overline{\psi}_0^{(s)}$

$$\begin{aligned}&\frac{\partial^3\overline{\psi}_0^{(s)}}{\partial\overline{\eta}^3} - \frac{\partial^2\overline{\psi}_0^{(s)}}{\partial\mu\partial\overline{\eta}}\frac{\partial\overline{\psi}_0^{(s)}}{\partial\overline{\eta}} + \frac{\partial\psi_0^{(s)}}{\partial\mu}\frac{\partial^2\overline{\psi}_0^{(s)}}{\partial\overline{\eta}^2} - \frac{\mu}{1-\mu^2}\left(\frac{\partial\overline{\psi}_0^{(s)}}{\partial\overline{\eta}}\right)^2\\
&= -\frac{\mu}{2}(1-\mu^2)\{(\overline{\Im}_0^{(s)})^2 + N(\overline{\Phi}_0^{(s)})^2\}\end{aligned} \tag{80}$$

Eq. (80) for stream function $\overline{\psi}_0^{(s)}$ is to be solved together with Eqs. (75) and (76) for the steady temperature $\overline{\Im}_0^{(s)}$ and concentration $\overline{\Phi}_0^{(s)}$ subject to the following boundary conditions

$$\overline{\psi}_0^{(s)} = \frac{\partial\overline{\psi}_0^{(s)}}{\partial\overline{\eta}} = 0, \quad \overline{\Im}_0^{(s)} = 1, \quad \overline{\Phi}_0^{(s)} = 1 \quad \text{on } \overline{\eta} = 0$$
$$\frac{\partial\overline{\psi}_0^{(s)}}{\partial\overline{\eta}} \to 0, \quad \overline{\Im}_0^{(s)} \to 0, \quad \overline{\Phi}_0^{(s)} \to 0 \quad \text{as } \overline{\eta} \to \infty \tag{81}$$

We notice again the embodiment of Reynolds stresses and buoyancy in the effective body-force term in Eq. (80). These terms make Eqs. (75), (76) and (80) completely different from the equations which describe the classical problem of steady free convection from a sphere immersed in a viscous fluid, when the buoyancy due to the mass diffusion and g-jitter effects are absent, see [20].

To start the numerical solution, we need to determine initial conditions for Eqs. (75), (76) and (80). To do this, we notice that the solution develops a singularity in the vicinity of $\mu = 1$ ($\theta = 0°$), i.e. at the pole of the sphere. Thus, we start the numerical solution near $\theta = 90°$, that is, at small values of μ and expand the functions $\overline{\psi}_0^{(s)}$, $\overline{\Im}_0^{(s)}$ and $\overline{\Phi}_0^{(s)}$ in the series of small μ of the form

$$\overline{\psi}_0^{(s)} = \mu f_0(\overline{\eta}) + \mathrm{O}(\mu^3), \quad \overline{\Im}_0^{(s)} = h_0(\overline{\eta}) + \mathrm{O}(\mu^2),$$
$$\overline{\Phi}_0^{(s)} = \phi_0(\overline{\eta}) + \mathrm{O}(\mu^2) \tag{82}$$

Substituting (82) into Eqs. (75), (76) and (80), we get the following ordinary differential equations for f_0, h_0 and ϕ_0

$$\begin{aligned}&f_0''' + f_0f_0'' - f_0'^2 = -\frac{1}{2}(h_0^2 + N\phi_0^2)\\
&h_0'' + Prf_0h_0' = 0, \quad \phi_0'' + Scf_0\phi_0' = 0\end{aligned} \tag{83}$$

subject to the boundary conditions (81), which become

$$f_0(0) = f_0'(0) = 0, \quad h_0(0) = 1, \quad \phi_0 = 1$$
$$f_0'(\infty) = 0, \quad h_0(\infty) = 0, \quad \phi_0(\infty) = 0 \tag{84}$$

where primes denote differentiation with respect to $\overline{\eta}$.

5. Results and discussion

The streamlines, $\psi_{01}^{(s)}$, of the steady flow at O(Re) given by Eq. (48) at equal intervals for the flow region outer to the Stokes ($Re \ll 1$) are shown in Fig. 1 for the buoyancy ratio parameter $N = 0$ and -5. We noticed from Eq. (48) that there is no flow for $N = -1$ because

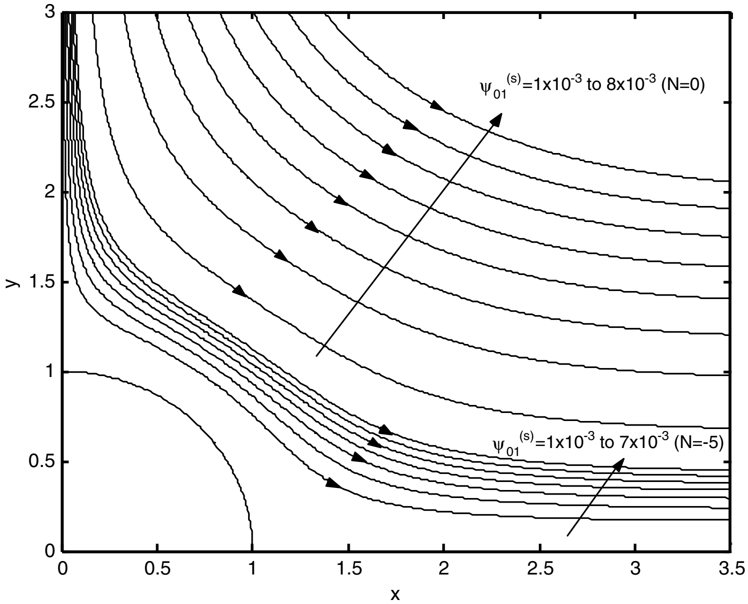

Fig. 1. Streamlines $\psi_{01}^{(s)}$ at equal intervals at O(Re) when $Re \ll 1$ for different values of N.

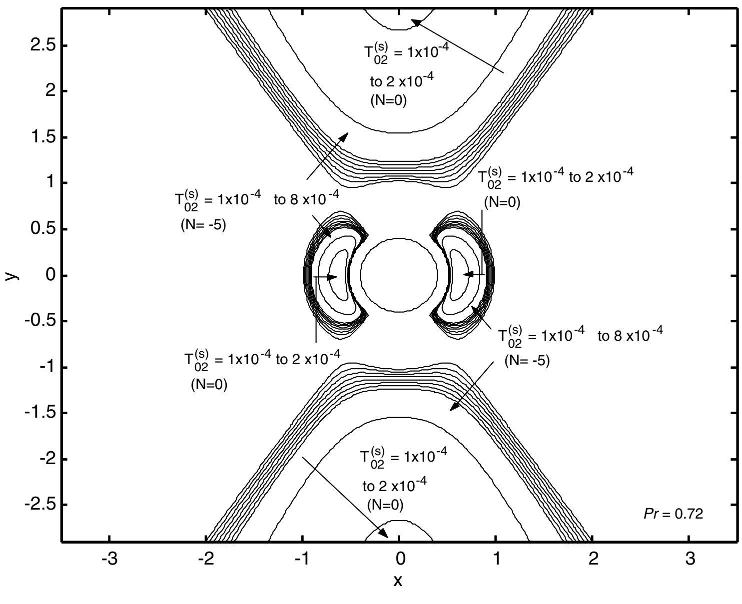

Fig. 2. Isothermal $T_{02}^{(s)}$ at equal intervals at O(Re^2) when $Re \ll 1$ for different values of N.

the two buoyancies are equal to and opposed each other. In addition the term $(1+N)^2$ in Eq. (48) takes the same values for $N=0,1,2,\ldots$ (assisting buoyant forces) and $N=-2,-3,\ldots$ (opposing buoyant forces), that means, the streamlines are the same for both assisting and opposing thermal and concentration buoyancies. Further, we can see from Fig. 1 that, as expected, fluid far from the sphere flows inwards perpendicular to the horizontal axis of the sphere and then outwards along its axis. Isotherms, $T_{02}^{(s)}$, and iso-concentration lines, $C_{02}^{(s)}$, given by Eqs. (59) and (60) at $O(Re^2)$ are illustrated in Figs. 2 and 3 for $N=0$ and $N=-5$ with $Pr=0.72$ (which represents air at 200 °C at 1 atm) and $Sc=1.6$ (which represents benzene, see [21]). It is seen that

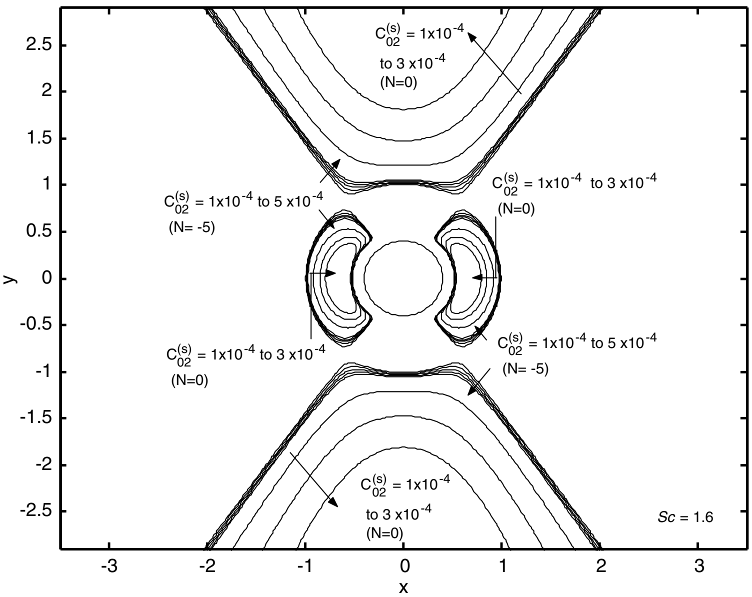

Fig. 3. Iso-concentration $C_{02}^{(s)}$ at equal intervals at $O(Re^2)$ when $Re \ll 1$ for different values of N.

Table 1
Values of reduced skin friction, local heat transfer and mass transfer for $Pr=0.72$ and $Sc=1.6$

θ	N								
	1			−0.5			−1		
	$\frac{\partial^2\overline{\psi}_0^{(s)}}{\partial\overline{\eta}^2}(\theta,0)$	$-\frac{\partial\overline{\Im}_0^{(s)}}{\partial\overline{\eta}}(\theta,0)$	$-\frac{\partial\overline{\Phi}_0^{(s)}}{\partial\overline{\eta}}(\theta,0)$	$\frac{\partial^2\overline{\psi}_0^{(s)}}{\partial\overline{\eta}^2}(\theta,0)$	$-\frac{\partial\overline{\Im}_0^{(s)}}{\partial\overline{\eta}}(\theta,0)$	$-\frac{\partial\overline{\Phi}_0^{(s)}}{\partial\overline{\eta}}(\theta,0)$	$\frac{\partial^2\overline{\psi}_0^{(s)}}{\partial\overline{\eta}^2}(\theta,0)$	$-\frac{\partial\overline{\Im}_0^{(s)}}{\partial\overline{\eta}}(\theta,0)$	$-\frac{\partial\overline{\Phi}_0^{(s)}}{\partial\overline{\eta}}(\theta,0)$
90°	0.672679	0.300956	0.457824	0.290155	0.247683	0.368943	0.121989	0.214096	0.311421
88°	0.302742	0.227703	0.395910	0.120503	0.182577	0.313540	0.082066	0.178244	0.282334
86°	0.183326	0.189491	0.354036	0.102532	0.168574	0.297716	0.069763	0.163782	0.268082
84°	0.243991	0.201168	0.362918	0.116771	0.170675	0.296854	0.063594	0.154989	0.258041
82°	0.183271	0.178703	0.334290	0.097112	0.157440	0.280586	0.060878	0.149658	0.250691
80°	0.256150	0.195682	0.349191	0.118373	0.164462	0.284592	0.060040	0.146506	0.245334
78°	0.211117	0.178116	0.324737	0.105616	0.155034	0.271556	0.060194	0.144400	0.240832
76°	0.283159	0.195451	0.341107	0.127643	0.163135	0.277277	0.061186	0.143372	0.237521
74°	0.243578	0.180148	0.319041	0.117295	0.155390	0.266042	0.062510	0.142762	0.234704
72°	0.314163	0.196626	0.334866	0.139142	0.163224	0.271667	0.063964	0.142356	0.232199
70°	0.274899	0.182304	0.314348	0.128957	0.156148	0.261570	0.065637	0.142167	0.229950
65°	0.359383	0.197831	0.325560	0.156086	0.163047	0.263462	0.068778	0.141052	0.224438
60°	0.331830	0.183553	0.302086	0.149907	0.155324	0.250522	0.070463	0.139033	0.218953
45°	0.374893	0.181147	0.292872	0.158752	0.147693	0.236269	0.066290	0.126723	0.200850
30°	0.270489	0.145196	0.256060	0.113845	0.118607	0.206882	0.047719	0.102842	0.176940
20°	0.127250	0.097528	0.204888	0.070706	0.090038	0.179576	0.029568	0.078931	0.154487
10°	0.073285	0.065228	0.172621	0.028621	0.052995	0.139409	0.010774	0.046468	0.120179
5°	0.037760	0.039015	0.134740	0.013384	0.031512	0.109184	0.003371	0.025657	0.091570

isotherms and iso-concentration lines are displayed symmetrically about the horizontal and vertical axes. However, the iso-concentration lines are closer to the sphere than the isotherms. This happens because when $Pr < Sc$ the thickness of the thermal boundary layer is larger than that of the concentration boundary layer, and, therefore, the wider thermal boundary layer drives the flow closer to the sphere.

Further, we have solved numerically the two systems of the steady-state boundary layer Eqs. (75), (76), (80) and (83) using a very efficient implicit finite-difference method known as the Keller-box method as described

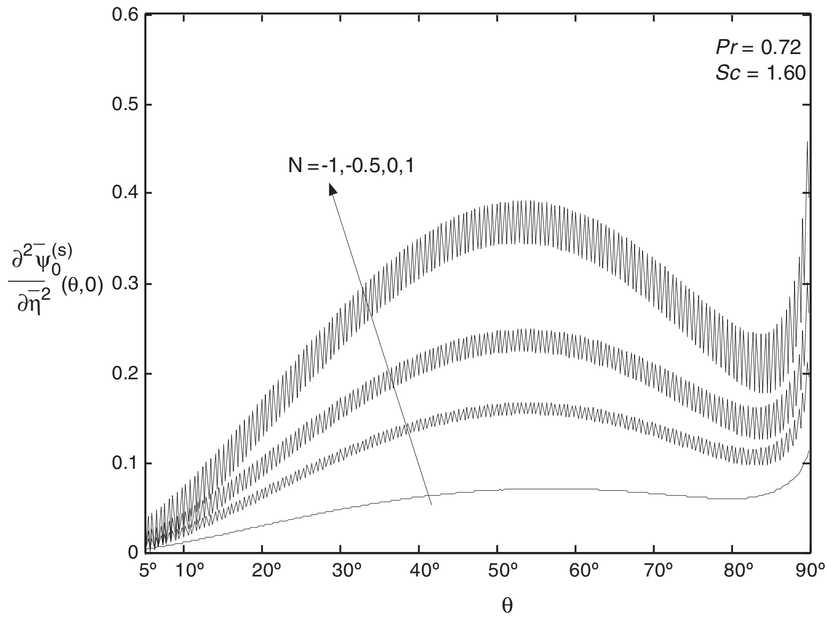

Fig. 4. Variations of skin friction with θ for different values of N for $Re \gg 1$.

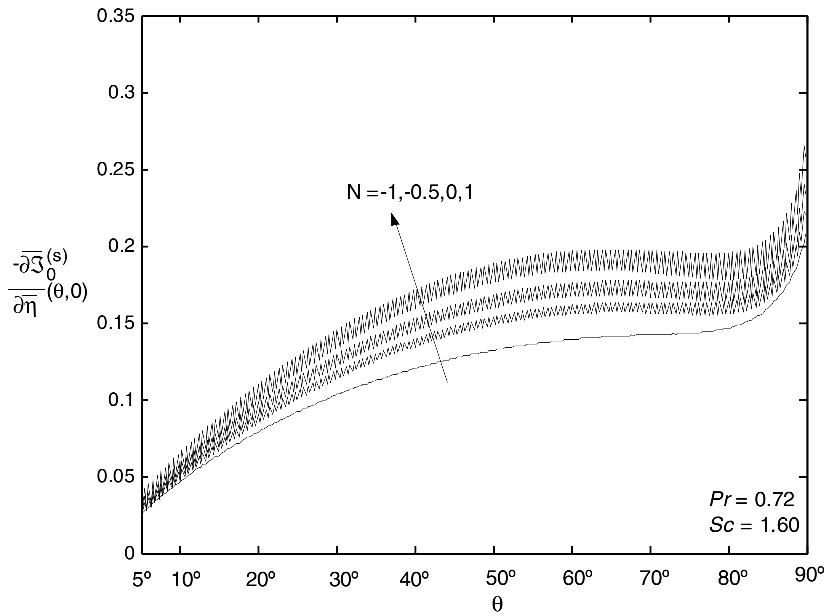

Fig. 5. Variations of heat flux with θ for different values of N for $Re \gg 1$.

in the book by Cebeci and Bradshaw [22] for the Prandtl $Pr = 0.72$ and 6.0, and Schmidt numbers $Sc = 1.6$ and 150, and at some positions θ around the sphere between $\theta = 0°$ and $\theta = 90°$ when $N = -1, -0.5, 0$ and 1. It is to be mentioned that the values of $Pr = 6.0$ and $Sc = 160$ have also been used by Mahajan and Angirasa [23] for the problem of steady combined heat and mass transfer by natural convection from a vertical flat plate. The quantities of primary interest in the present problem are the non-dimensional skin friction, $\partial^2\overline{\psi}_0^{(s)}(\theta,0)/\partial\overline{\eta}^2$, and the non-dimensional heat and mass transfer from the surface of the sphere, $-\partial\Im_0^{(s)}(\theta,0)/\partial\overline{\eta}$ and

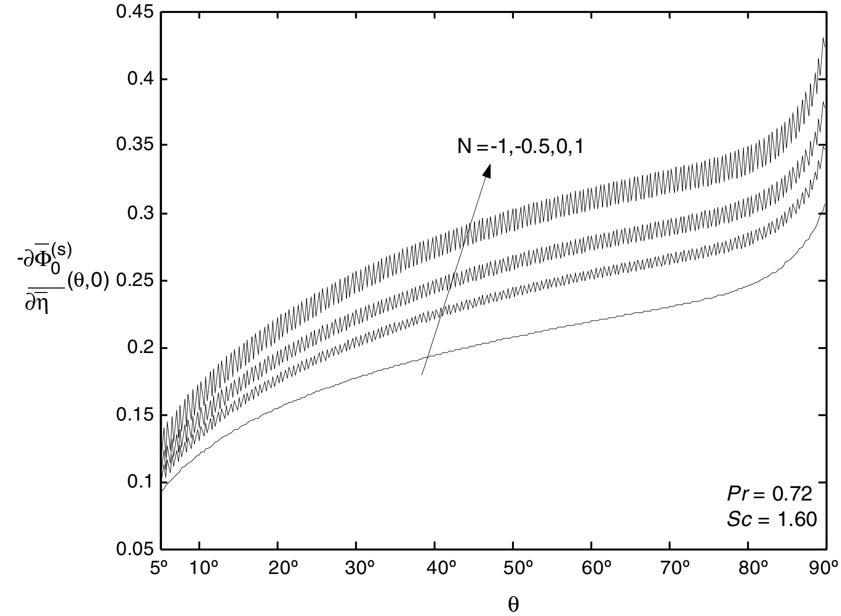

Fig. 6. Variations of concentration with θ for different values of N for $Re \gg 1$.

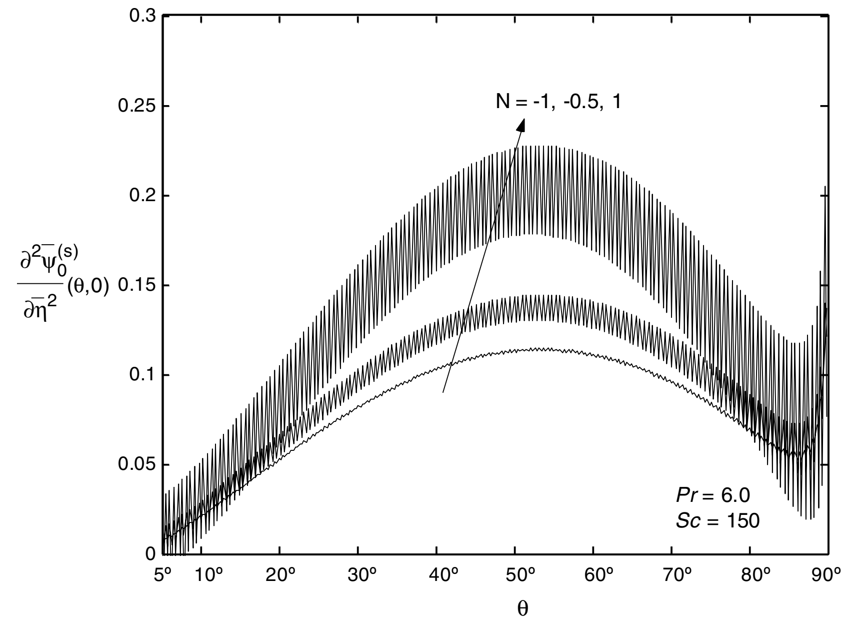

Fig. 7. Variations of skin friction with θ for different values of N for $Re \gg 1$.

$-\partial \Phi_0^{(s)}(\theta, 0)/\partial \overline{\eta}$, for the steady part of the solution induced by g-jitter. Some values of these quantities are given in Table 1 for $Pr = 0.72$ and $Sc = 1.6$, and several values of θ. The results for these quantities are also shown in Figs. 4–9. It is seen from both the table and figures that, as expected, skin friction and heat and mass transfer along the sphere have an oscillatory behaviour, which is attributed due g-jitter effects. These quantities are higher for the positive values of N (assisting buoyant forces) then for the negative values of N (opposing buoyant forces). However, the heat transfer parameter increase while the skin friction and mass transfer

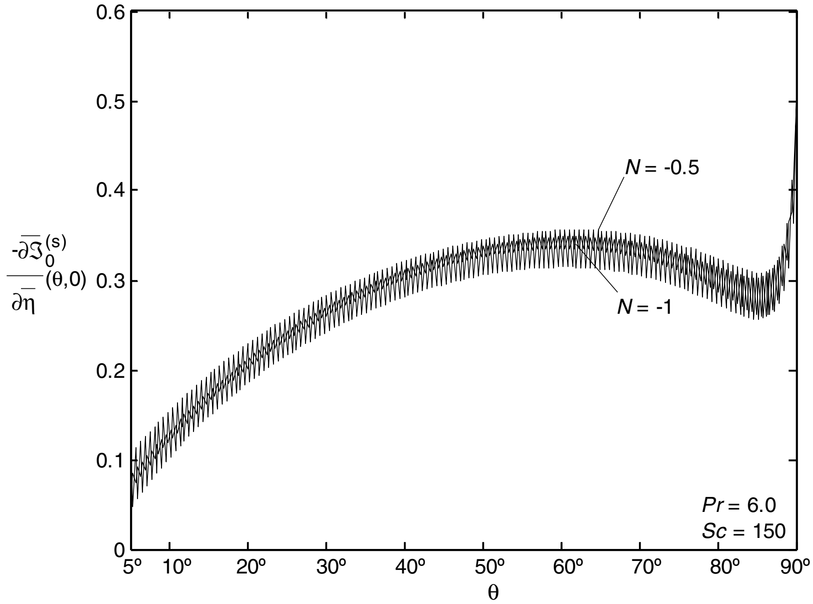

Fig. 8. Variations of heat flux with θ for different values of N for $Re \gg 1$.

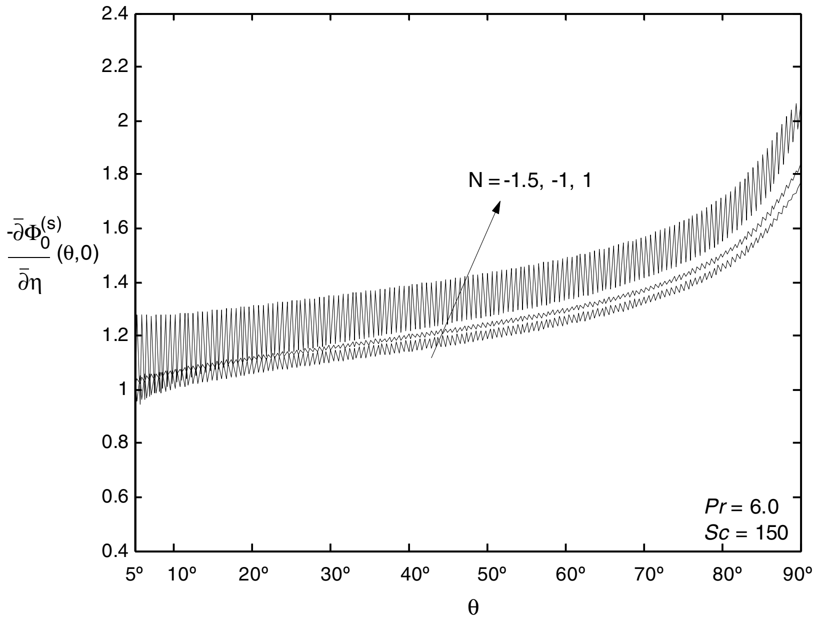

Fig. 9. Variations of concentration with θ for different values of N for $Re \gg 1$.

parameter decrease as both Pr and Sc increase. These figures show, in addition, that the mass transfer decreases almost continuously from the value of $\theta = 90°$ to a finite value at the upper pole ($\theta = 0°$). However, the skin friction and heat transfer parameters have minimum value close to $\theta = 90°$ and maximum values between $\theta = 50°$ and 70°. The peak of these profiles decreases as the values of Pr and Sc increase. Further, we notice from Figs. 4–9 that the skin friction decreases with the increase of Pr because a higher Pr implies more viscous fluid having a comparatively larger velocity (momentum) boundary thickness. But the heat transfer from the sphere increases with Pr. The physical reason for this trend is that a higher Prandtl number fluid has a thinner thermal boundary layer which increases the gradient of the temperature. Consequently the surface heat transfer from the sphere is increased as Pr increases. On the other hand, the mass concentration from the sphere decreases as both Pr and Sc numbers increase which means that the concentration boundary layer thickness decreases as both Pr and Sc increase. However, in these results, Sc ($= 1.6$ and 150) is much larger than Pr ($= 0.72$ and 6.0) and hence the concentration layer is much thinner than the thermal layer.

6. Conclusions

The method of inner and outer expansions is applied to the problem combined heat and mass transfer from a sphere which is subjected to a constant temperature and concentration and is immersed in a g-jitter gravity field with low and high Reynolds numbers and Prandtl and Schmidt numbers of order unity. The entire flow, temperature and concentration fields are determined to the second order in the small parameter of order $O(\varepsilon/Re)$. From the analysis we infer that the induced motion due to g-jitter oscillation has a steady part and an unsteady part, respectively. For $Re \gg 1$, the secondary steady motion, or acoustic streaming, consists of a momentum (velocity), thermal and concentration boundary layer, and the solution of the corresponding boundary layer equations is determined numerically using the Keller-box method. It is found that the skin friction increases indefinitely at the pole of the sphere ($\theta = 0°$), while heat and mass transfer remain finite at this point. It is worth mentioning that the solution presented here may also prove useful as a guide for more complex g-jitter accelerations such as, for example, a sum of Fourier harmonic components with distinct frequencies and amplitudes considered by Li [3]. It is hoped that the solution presented here, as well as in Amin [15], will further serve as a foundation for more complex and realistic studies of free and mixed convection flows from bodies of other configurations and also under the influence of an external magnetic field. It will also help to develop baselines for practical microgravity processing system design and development.

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